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Learning to Learn

The Art of Doing Science and Engineering

Session 14: Digital Filters I

Need to be mentally flexible!

Inability to update skills and education makes you an economic and social loss

- Economic loss because you cost more to employ than you are putting into the business
- Social loss because disgruntled employees foster an unhealthy work environment

You will have to learn a new subject many times during your career

- *Who Moved My Cheese*, http://www.whomovedmycheese.com

Digital Filters overview

Linear Processing implies digital filters

Theory dominated by Fourier Series

- Any complete set of functions (e.g. sinusoids) can do as well as any other set of arbitrary functions
- But why the almost-exclusive use of Fourier Series in field of digital signal processing?
  - *recent-year interest: wavelets*

What is really going on?

Digital Filters overview

Typically time-invariant representation of signals, given no natural origin of time.

Led to trigonometric functions, together with eigenfunctions of translation, in the form of Fourier series and Fourier integrals.

Linear systems use same eigenfunctions.

- Complex exponentials are equivalent to the real trigonometric functions

Digital Filters: Nyquist Sampling Theorem

Given a band-limited signal, sampled at equal spaces at a rate of at least two in the highest frequency, then the original signal can be reconstructed from the samples.

Sampling process loses no information when replacing continuous signal with equally spaced samples, provided that the samples can cover entire real number line.

Fourier Functions

Three reasons for using Fourier series:

- Time Invariance
- Linearity
- Reconstruction of the original function from the equally spaced samples

Nonrecursive Filters

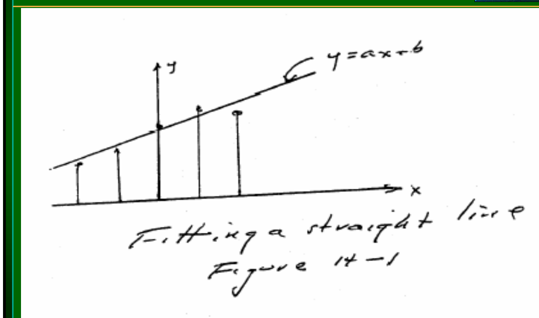
Sinusoidal Function

$$\cos at \cos bt = \frac{1}{2}(\cos(a+b)t + \cos(a-b)t)$$

Smoothing Type

$$y_n = \sum [j = -k, k; c_j u_{n-j}]$$

Figure 1



Nonrecursive Filters

Straight line to consecutive points of data

$$u(t) = a + bt$$
$$M = \sum [k = -2,2; \{u_k - (a + bk)\}^2]$$
$$-2\sum [u_k - a - bk] = 0$$
$$-2\sum [(uk - a - bk)k\} = 0$$
$$\sum [1] = 5 \qquad \sum [k^3] = 0$$
$$\sum [k] = 0 \qquad \sum [k^4] = 34$$
$$\sum [k^2] = 10$$

Nonrecursive Filters

$$\sum [u_k] = 5a + 0b$$
$$a = (1/5)\sum [k = -2,2; u_k]$$

Nonrecursive Filters

Smooth fitting quadratic equation

$$u(t) = a + bt + ct^2$$
$$-2\sum [\{u_k - a - bk - ck^2\}] = 0$$
$$-2\sum [\{u_k - a - bk - ck^2\}k] = 0$$
$$-2\sum [\{u_k - a - bk - ck^2\}k^2] = 0$$

Figure 2

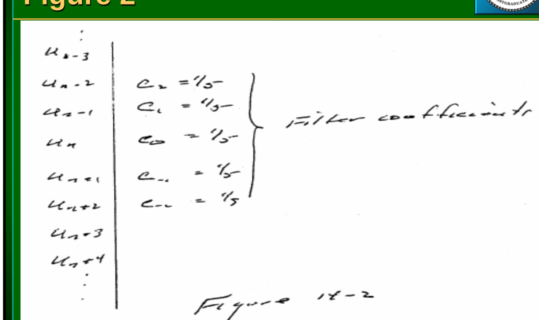
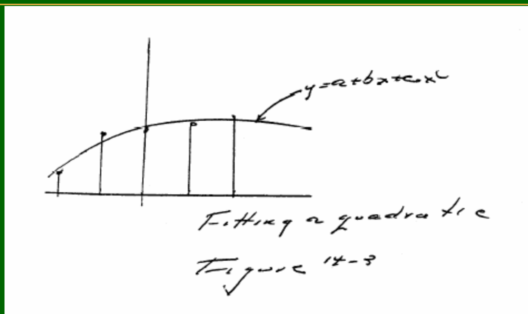


Figure 3



Nonrecursive Filters

$$5a + (10)c = \sum[u_k]$$
$$\{10\}a + \{34\}c = \sum[k^2 u_k]$$
$$\{85 - 50\}a = 17\sum[u_k] - 5\sum[k^2 u_k]$$
$$a = (1/35)[-3u_{-3} + 12u_{-2} + 17u_0 + 12u_1 - 3u_2]$$
$$u_n = (1/5)[u_{n-2} + u_{n-1} + u_n + u_{n+1} + u_{n+2}]$$
$$u_n = (1/35)[-3u_{n-2} + 12u_{n-1} + 17u_n + 12u_{n+1} - 3u_{n+2}]$$

Nonrecursive Filters

Pure Eigenfunction

$$u_n = (1/5)[\exp\{iw(n-1) + ... + \exp\{iw(n+2)\}$$
$$u_n = (1/5)e^{\{iwp\}}[e^{\{-2iw\}} + e^{\{-iw\}} + 1 + e^{\{iw\}} + e^{\{2iw\}}]$$

Transfer Function

$$H(w) = (1/5)\{2\cos 2w + 2\cos w + 1\}$$
$$H(w) = \{\sin(5/2)w\}/\{5\sin(1/2)w\}$$
$$H(w) = (1/35)\{17 + 24\cos w - 6\cos 2w\}$$

Figure 4

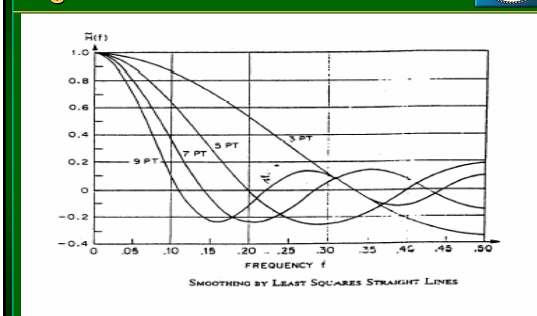
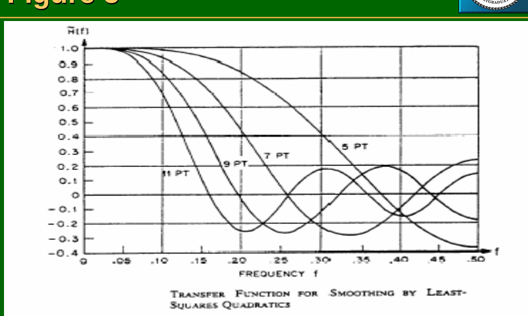


Figure 5



Nonrecursive Filters

Smoothing formulas have central symmetry in their coefficients, while differentiating formulas have odd symmetry.

$$f(x) = (1/2)[f(x) + f(-x) + (1/2[f(x) - f(x)]$$
$$f(t) = a_0/2 + \sum[k = 1, \infty; \{a_k \cos t + b_k \sin t\}]$$

Nonrecursive Filters

Orthogonality Conditions

$$\int_{-l}^{l} \cos kt \cos mt \, dt = \{0 \text{ for } k \neq m,\ l \text{ for } k = m \neq 0, 2l \text{ for } k = m = 0\}$$

$$\int_{-l}^{l} \cos kt \sin mt \, dt = 0 \text{ for all } m$$

$$\int_{-l}^{l} \sin kt \sin mt \, dt = \{0 \text{ for } k \neq m,\ l \text{ for } k = m \neq 0, 0 \text{ for } k = m = 0\}$$

$$\mathrm{a_k} = (1/l)\int [\mathrm{f(t)} \cos kt \, dt]$$

$$bk = (1/l)\int [\mathrm{f(t)} \sin kt \, dt]$$

Orthogonal Set

$$\int [w(t)\ f_k(t)\ f_m(t)]\, dt \text{ for } k \neq m, 1/l_k \text{ for } m = k$$

$$l_k = 1/\int [w(t)\, f_{k^2}(t)\, dt]$$

Orthogonal Set

$$\int [w(t)\ f(t)\, dt - \sum [\mathrm{c_k\ f_k(t)}]^2 \mathrm{dt} \geq 0$$

$$2\int [w(t)\{ f(t)dt - \sum c_k f_k(t)\} - f_k(t)dt] = 0$$

$$\int [w(t) f^2(t)dt - \sum [(1/l_k)c^2{}_k]] = \text{least squares error}$$