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Learning to Learn

The Art of Doing Science and Engineering

Session 16: Digital Filters III

Systematic Design of Non-Recursive Filters

Design Method

- Sketch an ideal filter
- Truncate the infinite Fourier series to 2N+1
- Remove the worst Gibb's Effect
- Observe Smoothed Function
- Weight the coefficients
- Reevaluate Fourier Series

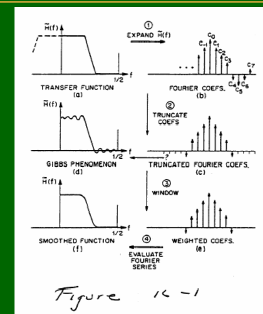


Figure 16-1

Amplification for Differentiation

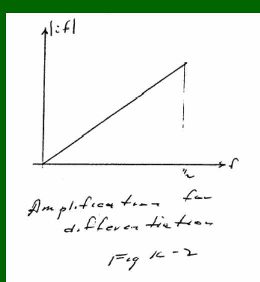


Fig 16-2

J.F. Kaiser Design Method

Finds both the N and the member of a family of windows to do the job.

- You have to specify two things beyond the shape:
  - *Vertical distance you are willing to tolerate missing the ideal.*
  - *Transition width between the pass and stop bands*

J.F. Kaiser Design Method

For a band pass filter, with f_p as the band pass and f_s as the band stop frequencies, the sequence of design formulas is:

$$A = -20\log_{10^d}$$

$$N \geq (A - 7.95)/28.72\Delta F \quad (N = \text{an integer})$$

If N is too big, reconsider your design, otherwise

$$a = 0.1102(A - 8.7) \qquad 50 < A$$

$$a = 0.5842(A - 21)0.4 + 0.7886(A - 21) \qquad 21 < A < 50$$

$$a = 0 \qquad A < 21$$

Band Pass Filters

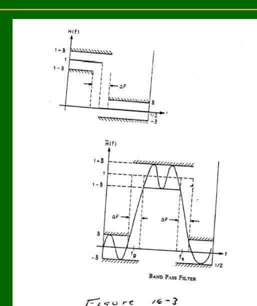
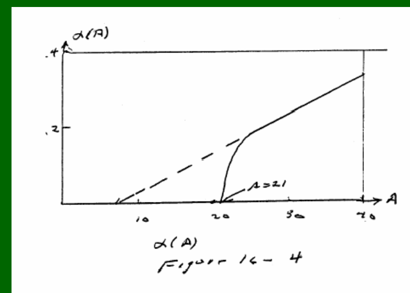


Figure 16-3

Original Fourier Coefficients for a Band Pass Filter

$$c_0 = 2(f_s - f_p)$$
$$c_k = (1/\,lk)[\sin 2\,lkf_s - \sin 2\,lk_p]\ \ (k = 1, 2, ..., N)$$
$$w_k = I_0[a\sqrt{(1-\{k/n\}^2)}\ ]/I_0(a)\ \ (|k| \le N), \text{ and 0 else}$$
where
$$\mathrm{I}_0(\mathrm{x}) = 1 + \sum_{n=1}^{\infty}[(x/2)/n!]^2$$
$$u_n = [(x/2)/n]^2 u_{n-1}$$

Bessel Function



Kaiser's Window Coefficient

At $k = 0 \quad w_0 = I_0(a)/I_0(a) = 1$

At $k = N \quad w_N = I_0(0)/I_0(a) = 1/I_0(a)$

For $a = 0$, we have something like the shape of a raised cosine

$a + b\cos x$

Resembles the vonHann and Hamming Windows for $A > 21$.

For $A < 21$ then $a = 0$, all the $w_k = 1$, resembling a Lanczos' type window.

How did Kaiser find the formulas?

First he assumed single discontinuity

He ran a large number of cases on the computer.

- As A increases he passed from a Lanczos' window to a platform of increasing height.
- Kaiser wanted a prolate spheroidal function but he noted they were accurately approximated.
- He plotted results and when one number, 0.5, didn't work, he dropped it to 0.4, and it did work
- Example of using what one knows plus the computer as an experimental tool to get very useful results.

Finite Fourier Series

The Fourier Functions are orthogonal, not only over a line segment, but for any discrete set of equally spaced points.

- Theory will be the same, expect that there can be only as many coefficients determined as there are points.
- Coefficients are determined as sums of the data points multiplied by the appropriate Fourier Functions.
- Resulting representation will, within roundoff, reproduce the original data.

Finite Fourier Series

Compute Expansion

- Compute by using 2N terms each with 2N multiplications and additions, ($2N^2$), operations of multiplication and addition.
- Using both:
  - *the addition and subtraction of terms with the same multiplier before doing the multiplication*
  - *Producing higher frequencies by multiplying lower ones, the Fast Fourier Transform (FFT).*

Finite Fourier Series

FFT has greatly transformed whole areas of science and engineering- what was once impossible in both time and cost is routinely done.

FFT and Tukey-Cooley paper.

- Moral of the Story- When you know that something cannot be done, also remember the essential reason why, so that later, when the circumstances have changed, you will not say, "It can't be done." When you decide something is not possible, don't say later that it is still impossible without reviewing all the details of why you originally were right in saying it couldn't be done.

Power Spectra

Which is the sum of the squares of the two coefficients of a given frequency in the real domain, or the square of the absolute value in the complex notation.

- Quantity does not depend on the origin of the time, but only on the signal itself, contrary to the dependence of the coefficients on the location of the origin.
- It was spectral lines that opened the black box of the atom and allowed Bohr to see inside.

Power Spectra

We regularly analyze black boxes by examining the spectrum of the input and the spectrum of the output, along with the correlations, to get an understanding of the insides - - not that there is always a unique inside, but generally we get enough clues to formulate a new theory.

Power Spectra

Let us analyze carefully what we do and its implications, because what we do to a great extent controls what we can see.

- Take a sample in time of length (2L).The original signal is convolved with the corresponding function of the form (sin x)/x.

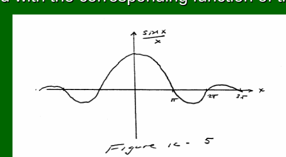


Figure 12 - 5

Power Spectra

Sample equal spaces in time, and all the higher frequencies are aliased to the lower frequencies.

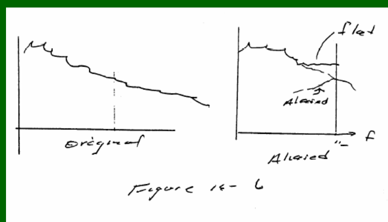
- Sampling and then limiting the range, will give the same results.
- When we assume the Finite Fourier Series representation we are making the function periodic.
- We force all non-harmonic frequencies into harmonic ones, we force a continuous spectrum to be a line spectrum.

Power Spectra

The spectrum of a sum of two signals is not the sum of a spectra.

- When you add two functions the individual frequencies are added algebraically, and they may happen to reinforce or cancel each other, and hence give entirely false results.
- Every spectrum of real noise falls off reasonably rapidly as you go to infinite frequencies.
- The sampling process aliases the higher frequencies into lower one, and the folding produces a flat spectrum.

Power Spectra



We call the flat spectrum for noise white noise. Noise is mainly in the lower frequencies.

Unstable- Stable

A bounded input if you are integrating could produce an unbounded output, which they said was unstable.

- But even a constant if integrated will produce a linear growth in the output.

Stability in digital filters means “not exponential growth” from bounded inputs, but allows polynomial growth, and this is not the standard stability criterion of classic analog filters.

Effects of Lanczos’ Window

Reduce Overshoot

- Reduced to 0.01189, a factor of 7
- Reduce first minimum to 0.00473, a factor of 10
- Significant but not a complete reduction of the Gibbs’ phenomenon.
- At discontinuity the truncated Fourier expansion takes on the mid-value of the two limits, one from each side.